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ON THE 'PITS EFFECT' OF LITTLEWOOD AND OFFORD

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ABSTRACT

Asymptotic behaviour of the entire functions $f(z) = \sum_{n=0}^{\infty} e^{2\pi i n \alpha_n} z^n/n!$, with real α_n is studied. It turns out that the Phragmén–Lindelöf indicator of such a function is always non-negative, unless $f(z) = e^{az}$. For a special choice of $\alpha_n = \alpha n^2$ with irrational α, the indicator is constant and f has completely regular growth in the sense of Levin and Pfluger. Similar functions of arbitrary order are also considered.

In [21] Nassif studied (at Littlewood's suggestion) the asymptotic behavior and the distribution of zeros of the entire function

$$\sum_{n=0}^{\infty} e^{2\pi i n^2 \alpha} z^n/n!, \tag{1}$$

with $\alpha = \sqrt{2}$. This work was continued by Littlewood [17, 18], who considered generalizations to Taylor series with coefficients that have smoothly varying moduli and arguments of the form $\exp(2\pi i \alpha n^2)$, where α is a quadratic irrationality.

Such functions behave similarly to random entire functions previously studied by Levy [16] and by Littlewood and Offord [19]; in particular, they display the 'pits effect', which Littlewood described as follows.

> If we erect an ordinate $|f(z)|$ at the point z of the z-plane, then the resulting surface is an exponentially rapidly rising bowl, approximately of revolution, with exponentially small pits going down to the bottom. The zeros of f, more generally the w-points where $f = w$, all lie in the pits for $|z| > R(w)$. Finally the pits are very uniformly distributed in direction, and as uniformly distributed in distance as is compatible with the order ρ.

The earliest study of functions (1) known to the authors is the thesis of Ålander [1], who considered the case of rational α. Levy [16] used the results of Hardy and Littlewood on Diophantine approximation to prove the following. Let

$$M(r, f) = \max_{|z|=r} |f(z)| \qquad \text{and} \qquad m_2^2(r, f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 d\theta.$$

Then

$$M(r, f)/m_2(r, f) \qquad \text{is bounded} \tag{2}$$

for f of the form (1), and α satisfying a Diophantine condition. This is even stronger regularity than random arguments of coefficients yield [16, 19]. Some other works where the function (1) with various α was studied or used are [7, 8, 20, 27].

Function (1) is the unique analytic solution of the functional equation

$$f'(z) = qf(q^2 z), \qquad \text{where } q = e^{2\pi i \alpha}, \text{ and } f(0) = 1, \tag{3}$$

which is a special case of the so-called 'pantograph equation'. There is considerable literature on this equation with real q; see, for example, [13, 14] and references there.

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Recently, there has been a renewed interest in the functions of the type (1) because they arise as the limits as $q \to e^{2\pi i\alpha}$ of the function of two variables

$$\sum_{n=0}^{\infty} q^{n^2} z^n/n!,$$

which plays an important role in graph theory [**26**] and statistical mechanics (as described in a private communication to the authors from Alan Sokal). This function is the unique solution of (3), for all q in the closed unit disc.

In the present paper, we first study arbitrary entire functions of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n/n!, \qquad \text{where } |a_n| = 1. \tag{4}$$

Our Theorem 1 says that such functions cannot decrease exponentially on any ray, unless f is an exponential. This can be compared with a result of Rubel and Stolarski [**23**], that there exist exactly five series of the form (4) with $a_0 = 0$, $a_n = \pm 1$, which are bounded on the negative ray. Our second result, Theorem 2 shows that one cannot replace the condition of exponential decrease in Theorem 1 by boundedness on a ray: there are infinitely many functions of the form (4) which tend to zero as $z \to \infty$ in the closed right half-plane.

In the second part of the paper, we consider the case $\arg a_n = 2\pi i n^2\alpha$ with *any irrational* α. Theorem 3 shows that the qualitative picture of $|f(z)|$ is the same as that described by Littlewood, except that our estimate of the size of the pits is worse than exponential. In particular, we show that

$$\log|f(z)| = |z| + o(|z|),$$

outside some exceptional set of z. According to the Levin–Pfluger theory [**15**], this behavior of $|f|$ has the following consequences about the zeros z_k of f.

The number $n(r, \theta_1, \theta_2)$ of zeros (counting multiplicity) in the sector

$$\{z : \theta_1 < \arg z < \theta_2,\ |z| < r\}$$

satisfies

$$n(r, \theta_1, \theta_2) = \frac{\theta_2 - \theta_1}{2\pi}(r + o(r)) \qquad \text{as } r \to \infty. \tag{5}$$

Moreover, the limit

$$\lim_{R\to\infty} \sum_{|z_k|\leqslant R} \frac{1}{z_k} \tag{6}$$

exists, where z_k is the sequence of zeros of f. It is easy to see from the Taylor series of f that this limit equals $-q$.

Thus the Diophantine conditions used in [**16**, **21**, **27**] are unnecessary for the qualitative picture of the behavior of $|f|$, but with arbitrary irrational α the results are less precise then those where α satisfies a Diophantine condition. Theorem 4 shows that Levy's result (2) cannot be extended to arbitrary irrational α. Finally, we prove a result similar to Theorem 3 where the condition $|a_n| = 1$ is replaced by a more flexible condition on the moduli of the coefficients, allowing the function to have any order of growth.

We denote by

$$F(z) = \sum_{n=1}^{\infty} a_{n-1} z^{-n},$$

the Borel transform of f in (4) (using the terminology of [**15**]). Then F has an analytic continuation from a neighborhood of infinity to the region $\overline{\mathbb{C}} \setminus K$, where K is a convex compact set in the plane, which is called the conjugate indicator diagram.

The indicator

$$h_f(\theta) := \limsup_{r\to\infty} r^{-1} \log |f(re^{i\theta})|, \qquad |\theta| \leqslant \pi,$$

is the support function of the convex set symmetric to K with respect to the real axis.

We also consider the function

$$G(z) = \sum_{n=1}^{\infty} a_{n-1} z^n$$

analytic in the unit disc. Transition from F to G is by the change of the variable $1/z$.

Pólya's theorem [15, Appendix I, Section 5]. *Suppose that G has an analytic continuation from the unit disc to infinity through some angle $|\arg z - \pi| < \delta$. Then the coefficients a_n can be interpolated by an entire function g of exponential type such that the indicator diagram of g is contained in the horizontal strip $|\Im z| \leqslant \pi - \delta$. That is, $g(n) = a_n$ for $n \geqslant 1$, and $h_g(\pm\pi/2) \leqslant \pi - \delta$.*

Carlson's theorem [15, Chapter IV, Introduction]. *Suppose that the indicator diagram of an entire function g has width less than 2π in the direction of the imaginary axis; that is, $h_g(\pi/2) + h_g(-\pi/2) < 2\pi$. Then g cannot vanish on the positive integers, unless $g = 0$.*

Theorem 1. *Every entire function f of the form (4) has non-negative indicator, unless $a_n = \mathrm{const} \cdot a^n$ for some a on the unit circle, in which case $f(z) = e^{az}$.*

By Borel's transform, this is equivalent to the following theorem.

Theorem 1′. *Let G be as above. Then G cannot have an analytic continuation to infinity through any half-plane containing 0, unless $a_n = \mathrm{const} \cdot a^n$ for some a.*

These two theorems give characterizations of the exponential function and the geometric series, respectively, showing that their behavior is quite exceptional. Another somewhat similar characterization follows from the result in [23] mentioned above. As a corollary from Theorem 1 we obtain the following result of Carlson [4]: if z_n is the sequence of zeros of f as in (4), then

$$\sum_n \frac{1}{|z_n|} = \infty, \tag{7}$$

unless f is an exponential.

Proof of Theorem 1′. Suppose that G has such an analytic continuation. Replacing z by az with $|a| = 1$ we ensure that G has an analytic continuation to infinity through some left half-plane of the form $\Re z < \varepsilon$, where $\varepsilon > 0$.

Pólya's theorem then implies that $a_n = g(n)$ for some entire function for which the indicator diagram is contained in the strip $|\Im z| < \pi/2 - \delta$, for some $\delta > 0$. Consider the functions

$$g_R(z) = \frac{g(z) + \overline{g(\overline{z})}}{2} \qquad \text{and} \qquad g_I(z) = \frac{g(z) - \overline{g(\overline{z})}}{2i}.$$

On the real axis we have $g_R(x) = \Re g(x)$ and $g_I(x) = \Im g(x)$. Consider the entire function

$$H = g_I^2 + g_R^2.$$

Then at positive integers we have

$$H(n) = g_I^2(n) + g_R^2(n) = (\Re g(n))^2 + (\Im g(n))^2 = |a_n|^2 = 1.$$

So the function $H-1$ has zeros at all positive integers. Its indicator diagram is contained in the strip

$$|\Im z| \leqslant \pi - 2\delta.$$

(Squaring stretches the indicator diagram by a factor of 2, and the indicator diagram of the sum of two functions is contained in the convex hull of the union of their diagrams.) Now, by Carlson's theorem, $H \equiv 1$, so

$$g_I^2 + g_R^2 = 1. \tag{8}$$

The general solution of this functional equation in the class of entire functions is $g_I = \cos\circ\phi$, $g_R = \sin\circ\phi$, where ϕ is an entire function. It is well known and easy to see that for g_I and g_R to be of exponential type, it is necessary and sufficient that $\phi(z) = cz + b$. As g_I and g_R are real on the real line, we conclude that c and b are real. Thus $a_n = \cos(cn+b) + i\sin(cn+b) = \text{const}\cdot e^{icn} = \text{const}\cdot a^n$, as stated.

An alternative way to derive the conclusion from (8), suggested by Katsnelson, is to notice that (8) implies that

$$|g(x)| \equiv 1 \qquad \text{for real } x. \tag{9}$$

The symmetry principle then implies that g has no zeros. (If z_0 were a zero, then $\overline{z}_0$ would be a pole.) So g is a function of exponential type without zeros, and thus $g = \exp(icz)$, where c should be real by (9). □

As we have already noted, Theorem 1 implies (7). However, it does not imply that the sequence of zeros has positive density: there exist functions of exponential type, even with constant indicator, with zeros that have zero density. (We note that Valiron [**27**, p. 415] erroneously asserted the contrary: that for functions with constant indicator, zero cannot be a Borel exceptional value.) To construct such examples, take zeros of the form

$$z_k = \left(e^{i\log\log(k+1)} - e^{i\log\log k}\right)^{-1}, \qquad k \geqslant 2,$$

and construct the canonical product W of genus one with such zeros. It is not hard to show that the asymptotic behavior of this product will be

$$\log|W(re^{i\theta})| = (cr + o(r))\sin(\theta - \log\log r), \qquad r \to \infty,$$

outside of some small exceptional set, so the indicator h_W is constant, while the density of zeros is zero: $|z_k| \sim k\log k$, $k \to \infty$.

There exist entire functions of the form (4), other than the exponential, which are bounded in the left half-plane. The simplest example is Hardy's generalization of e^z defined by the power series

$$E_{s,a} = \sum_{n=1}^{\infty} \frac{(n+a)^s z^n}{n!}, \qquad s \in \mathbb{C},\ a > 0.$$

For pure imaginary s, this series is of the form (4). Hardy [**10**] proved the asymptotic formula

$$E_{s,a}(z) = z^s e^z(1+o(1)) + \frac{\Gamma(a)}{\Gamma(-s)(-z)^a\log(-z)}(1+o(1)),$$

as $z \to \infty$, $|\arg z \pm \pi/2| < \varepsilon$, for every $\varepsilon \in (0, \pi/2)$. This formula implies that the functions $E_{s,a}$ with pure imaginary s are bounded in the closed left half-plane. For further results on Hardy's function, see [**22**].

THEOREM 2. *Let ψ be a real entire function with the property*

$$\psi(\zeta) = o(|\zeta|), \qquad \zeta \to \infty$$

in every half-plane $\Re\zeta > c$, $c \in \mathbb{R}$. Then the function

$$f(z) = \sum_{n=0}^{\infty} \frac{e^{i\psi(n)}}{n!} z^n$$

is of the form (4) and for every $A > 0$ and every $\varepsilon > 0$ we have

$$|f(re^{i\phi})| = O(r^{-A}), \qquad r \to \infty, \tag{10}$$

uniformly for $|\phi - \pi| \leqslant \pi/2 - \varepsilon$.

Proof. We have the following integral representation:

$$\begin{aligned} f(-z) &= -\frac{1}{2\pi i} \int_{-A-i\infty}^{-A+i\infty} \frac{\pi e^{i\psi(\zeta)} z^{\zeta}}{\Gamma(\zeta+1)\sin \pi\zeta}\, d\zeta, \\ &= \frac{1}{2\pi i} \int_{-A-i\infty}^{-A+i\infty} e^{i\psi(\zeta)} z^{\zeta} \Gamma(-\zeta)\, d\zeta, \end{aligned} \tag{11}$$

where $A > 0$ is any positive number. To obtain this representation, we notice that that by Stirling’s formula, the modulus of the integrand does not exceed

$$|z|^{-\Re\zeta} \exp\left((-\pi/2 + \phi + o(1))|\Im\zeta|\right),$$

as $|\zeta| \to \infty$ in every half-plane of the form $\Re\zeta \geqslant -A$. Here $o(1)$ is independent of z. Applying the residue formula to the rectangle

$$\{\zeta : -c < \Re\zeta < N + 1/2,\ |\Im\zeta| < N + 1/2\},$$

and letting N tend to infinity, we obtain (11). Now the same estimate of the integrand shows that (10) holds. □

Theorem 1 implies that the indicator diagram of a function of the form (4), other than an exponential, contains zero. Theorem 2 shows that the indicator diagram of such a function can be contained in a closed half-plane. It seems interesting to describe all possible indicator diagrams that can occur for functions of the form (4). We have the following partial result.

PROPOSITION. *For an arbitrary finite set Z on the unit circle, there exists an entire function of the form (4), and for which the indicator diagram coincides with the convex hull of $Z \cup -Z$.*

Proof. Let E be the set of all entire functions of the form 4. We consider the following operators on E:

$$\begin{aligned} R_\theta[f](z) &:= f(ze^{-i\theta}), \\ C[f](z) &:= \tfrac{1}{2}\left(f(z) + f(-z)\right), \end{aligned}$$

and

$$S[f] := \tfrac{1}{2}\left(f(z) + f(-z)\right).$$

Now we define an operator $E \times E \to E$ by the formula

$$Q_{\theta_1,\theta_2}[f_1, \ldots, f_2] = (C \circ R_{\theta_1})[f_1] + (S \circ R_{\theta_2})[f_2].$$

It can be easily shown that if $f \in E$ is a function with indicator diagram $[0, 1]$, then $(C \circ R_\theta)[f]$ and $(S \circ R_\theta)[f]$ have indicator diagram $[-e^{i\theta}, e^{i\theta}]$. Hence the indicator diagram of $f_1 = Q_{\theta_1,\theta_2}[f,f]$ is the convex hull of

$$\{e^{i\theta_1}, -e^{i\theta_1}, e^{i\theta_2}, -e^{i\theta_2}\}.$$

This proves the proposition for the sets Z of two points. Then we consider $f_2 = Q_{0,\theta_3}[f_1, f]$, and so on. $\square$

Now we consider functions of the form (4) with $\arg a_n = 2\pi n^2\alpha$, $\alpha \in \mathbb{R}$.

THEOREM 3. *Let f be of the form (4) with $a_n = \exp(2\pi i n^2\alpha)$, where α is irrational. Then f has completely regular growth in the sense of Levin and Pfluger, and $h_f \equiv 1$.*

We recall the main facts of the Levin–Pfluger theory in modern language [**2**]. Fix a positive number ρ. Let u be a subharmonic function in the plane satisfying

$$u(z) \leqslant O(r^\rho), \qquad r \to \infty.$$

Then the family of subharmonic functions

$$A_t u(z) = t^{-\rho} u(tz), \qquad t > 1,$$

is bounded from above on every compact subset of the plane. Such families of subharmonic functions are pre-compact in the topology D' of Schwartz's distributions [**11**, Theorem 4.1.9], so from every sequence $A_{t_k}u$, $t_k \to \infty$ one can select a convergent subsequence. An entire function f or order ρ, normal type, is called of *completely regular growth* if the limit

$$u = \lim_{t\to\infty} A_t \log|f| \tag{12}$$

exists. It is easy to see that this limit is a fixed point for all operators A_t, so it has the form

$$u(re^{i\theta}) = r^\rho h(\theta),$$

and h is the indicator of f. Operators A_t also act on measures in the plane by the formula

$$A_t\mu(E) = t^{-\rho}\mu(tE), \qquad \text{for } E \subset \mathbb{C}.$$

The Riesz measure μ_f of $\log|f|$ is the counting measure of zeros of f, and one of the results of Levin and Pfluger can be stated as follows: the existence of the limit (12) implies the existence of the limit

$$\mu = \lim_{t\to\infty} A_t\mu_f.$$

This limit μ is also fixed by all operators A_t, so

$$d\mu = r^{\rho-1}\, dr\, d\nu(\theta),$$

where ν is a measure on the unit circle which is called the *angular density* of zeros. This measure ν is related to the indicator by the formula

$$d\nu = (h'' + \rho^2 h)\, d\theta,$$

in the sense of distributions.

Thus, as a corollary of Theorem 3, we find that the angular density of zeros of f is a constant multiple of the Lebesgue measure.

Completely regular growth with indicator 1 and order $\rho = 1$ implies that

$$\log|f(re^{i\theta})| = r + o(r) \qquad \text{as } r \to \infty, \tag{13}$$

uniformly with respect to θ, when $re^{i\theta}$ does not belong to an exceptional set. According to Azarin [**2**], for every $\eta > 0$ this exceptional set can be covered by discs centered at w_k and of radii r_k such that

$$\sum_{|w_k|\leqslant r} r_k^{\eta} = o(r^{\eta}), \qquad r\to\infty. \tag{14}$$

This improves the original condition with $\eta = 1$ given in [**15**]. The properties (5) and (6) of zeros of f, stated in the beginning of the paper, follow from (13) by [**15**, Theorems II.2 and III.4]; see also [**24**].

The exceptional set (14) is larger than the exceptional set in the work of Nassif. The exceptional set in Theorem 3 could be improved to a set of exponentially small circles if one knew that the zeros of f were well separated. This seems to be an interesting unsolved problem about the function (1). In particular, *can f of the form (1) have a multiple zero*? For $\alpha = \sqrt{2}$, Nassif proved that all but finitely many zeros are simple and well separated.

That the indicator of f in Theorem 3 is constant was proved by Valiron [**27**, p. 412]. (Valiron obtained an equation which is equivalent to our (20) below [**27**, equation (11)], but he did not fully explore its consequences. Later in the same paper [**27**, p. 421], Valiron proves that f is of completely regular growth only under an additional Diophantine condition on α.) This also follows from the result of Cooper [**6**], who proved that the corresponding function G has the unit circle as its natural boundary, see also [**5**, p. 76, Footnote] where a short proof of Cooper's theorem is given. However, as we noticed above, constancy of the indicator by itself only implies (7); it is the statement about completely regular growth that permits us to conclude that the zeros have positive density.

Proof of Theorem 3. By differentiating the power series, it is easy to obtain

$$f'(z) = e^{2\pi i\alpha} f(ze^{i\beta}), \qquad \text{where } \beta = 4\pi\alpha. \tag{15}$$

(This is the 'pantograph equation' (3) with $q = e^{2\pi i\alpha}$.) The assumption that $|a_n| = 1$ implies the following behavior of $M(r,f)$:

$$\log M(r,f) = r + o(r). \tag{16}$$

This is proved by the standard argument relating the growth of $M(r,f)$ with the moduli of the coefficients; see, for example [**15**, Chapter I, Section 2]. The bounds $0 \leqslant r - \log M(r,f) \leqslant (1/4 + o(1))\log r$ can be obtained as follows. The upper bound $M(r,f) \leqslant e^r$ is trivial, and for the lower bound, we use Cauchy's inequality $M(r,f) \geqslant r^n/n!$, and maximize the right-hand side with respect to n. In particular, the order $\rho = 1$.

It follows from (16) that the family of subharmonic functions

$$\{u_t = A_t \log|f| : 0 < t < \infty\}$$

is uniformly bounded from above on compact subsets of $\mathbb{C}$. Moreover, $u_t(0) = 0$. So every sequence $\sigma = (t_k) \to \infty$ contains a subsequence σ' such that the limit

$$u = \lim_{t\in\sigma',\, t\to\infty} u_t \tag{17}$$

exists in D', the space of Schwartz's distributions in the plane. The set of all possible limits u for all sequences σ is called the limit set of f, and is denoted by $\mathrm{Fr}[f]$. It consists of subharmonic functions in the plane satisfying $u(0) = 0$. Equation (16) implies that

$$\max_{|z|\leqslant r} u(z) = r, \qquad 0 \leqslant r < \infty. \tag{18}$$

If $u = \lim t_k^{-1}\log|f(t_k z)|$, and $v = \lim t_k^{-1}\log|f'(t_k z)|$ with the same sequence $t_k\to\infty$, then

$$v \leqslant u. \tag{19}$$

Indeed, by Cauchy's inequality, for every $\varepsilon > 0$ and $|z| > 1/\varepsilon$, we have

$$\log|f'(z)| \leqslant \max_{|\zeta|\leqslant\varepsilon|z|} \log|f(z+\zeta)|.$$

This implies that for every $\varepsilon > 0$,

$$v(z) \leqslant \max_{|\zeta|\leqslant\varepsilon} u(z+\zeta).$$

Now the upper semi-continuity of subharmonic functions shows that the right-hand side of the last equation tends to $u(z)$ as $\varepsilon \to 0$, which proves (19).

The functional equations (15) and (19) imply that $u(ze^{i\beta}) \leqslant u(z)$, and this gives

$$u(ze^{i\beta}) \equiv u(z). \tag{20}$$

As β is irrational, $u(z)$ is independent of $\arg z$, and taking (18) into account we conclude that the limit set $\mathrm{Fr}[f]$ consists of the single function $u(z) = |z|$. This means that f is of completely regular growth, with constant indicator. □

Now we show that there exist irrational α such that the corresponding functions f_α in (1) do not have property (2).

THEOREM 4. *There is a residual set E on the unit circle, such that for a function f_α as in (1) with $\alpha \in E$, we have*

$$\limsup_{r\to\infty} \frac{M(r,f)}{m_2(r,f)} = \infty. \tag{21}$$

We recall that a set is called *residual* if it is an intersection of countably many dense open sets. By Baire's category theorem, residual sets on $[0,1]$ have the power of a continuum, and thus contain irrational points.

Proof of Theorem 4. Consider the sets

$$E_{m,n} = \left\{\alpha : \frac{M(r,f_\alpha)}{m_2(r,f_\alpha)} \leqslant m \text{ for } r \geqslant n\right\},$$

where m and n are positive integers. Evidently, all these sets are closed. Let $E = [0,1] \setminus \bigcup_{m,n} E_{m,n}$. Then for $\alpha \in E$ we have (21), and E is a countable intersection of open sets. It remains to show that E is dense. We will show that E contains all rational numbers. Indeed, for rational α, f_α is a finite trigonometric sum:

$$f_\alpha = \sum c_k e^{b_k z}, \tag{22}$$

where b_k are roots of unity. This representation follows immediately from the functional equation (15): iterating this functional equation finitely many times, we obtain a linear differential equation with solutions that are trigonometric sums (22). It is clear that any finite trigonometric sum g satisfies

$$M(r,g)/m_2(r,g) \to \infty \qquad \text{as } r \to \infty.$$

This proves that E is dense and thus residual. □

Now we extend Theorem 3 to the case where $|a_n|$ is non-constant. For this we need Hadamard's multiplication theorem, as follows.

Hadamard's multiplication theorem [3]. *Let*

$$f = \sum_{n=0}^{\infty} c_n z^n$$

be an entire function, and let

$$H = \sum_{n=0}^{\infty} b_n z^n$$

be a function analytic in $\overline{\mathbb{C}} \setminus \{1\}$. *Then the function*

$$(f \star H)(z) = \sum_{n=0}^{\infty} a_n c_n z^n$$

has the integral representation

$$(f \star H)(z) = \frac{1}{2\pi i} \int_C f(\zeta) H\left(\frac{z}{\zeta}\right) \frac{d\zeta}{\zeta},$$

where C *is any closed contour going once in positive direction around the point* 1.

This operation $f \star H$ is called the *Hadamard composition of power series*. From the integral representation we immediately obtain

$$|(f \star H)(z)| \leqslant K \max_{|\zeta - z| \leqslant r|z|} |f(\zeta)|, \qquad \text{where } K = \max_{|\zeta - 1| = r/(1-r)} |H(\zeta)|. \tag{23}$$

Theorem 5. *Let* h *be an entire function of minimal exponential type. Let*

$$c_n = h(0)h(1)\dots h(n), \qquad n \geqslant 0, \tag{24}$$

and assume in addition that

$$-\log|c_n| = \frac{1}{\rho} n \log n - cn + o(n), \qquad n \to \infty, \tag{25}$$

with some real constant c. *Then the entire function*

$$f(z) = \sum_{n=0}^{\infty} c_n e^{2\pi i n^2 \alpha} z^n,$$

with irrational α, *has order* ρ, *normal type and completely regular growth with constant indicator.*

The condition that c_n/c_{n-1} is interpolated by an entire function of minimal exponential type is not as rigid as it may seem. In this connection, we recall a theorem of Keldysh (see, for example, [9]) that every function h_1 analytic in the sector $|\arg z| < \pi - \varepsilon$ and satisfying $\log|h_1(z)| = O(|z|^\lambda)$, $z \to \infty$ there, with $\lambda = \pi/(\pi + \varepsilon) < 1$, can be approximated by an entire function h of normal type, order λ, so that

$$|h(z) - h_1(z)| = O(e^{-|z|^\lambda}), \qquad z \to \infty,\ |\arg z| < \pi - 2\varepsilon.$$

For example, one can take

$$h_1(z) = z^{-1/\rho}, \qquad \rho > 0,$$

and apply Keldysh's theorem, to obtain a function f of normal type, order ρ, satisfying all the conditions of Theorem 5.

Proof of Theorem 5. We write:

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} c_n e^{2\pi i n^2\alpha} z^n \\ &= 1 + \sum_{n=0}^{\infty} c_{n+1} e^{2\pi i (n+1)^2\alpha} z^{n+1} \\ &= 1 + ze^{2\pi i\alpha} \sum_{n=0}^{\infty} c_{n+1} e^{2\pi i n^2\alpha} \left(ze^{4\pi i n\alpha}\right)^n \\ &= 1 + ze^{2\pi i\alpha}(f \star H)(ze^{i\beta}), \quad \text{where } \beta = 4\pi\alpha, \end{aligned}$$

and

$$H(z) = \sum_{n=0}^{\infty} \frac{c_{n+1}}{c_n} z^n.$$

By assumption (24) of the theorem, and Pólya's theorem above, H is holomorphic in $\overline{\mathbb{C}} \setminus \{1\}$, so the estimate (23) holds. Assumption (24) implies that

$$M(r, f) = \sigma r^\rho + o(r^\rho), \qquad r \to \infty,$$

where $\sigma = e^{c\rho}/(e\rho)$; see [**15**, Chapter I, Section 2]. So from every sequence one can select a subsequence such that the limits

$$u(z) = \lim_{k\to\infty} t_k^{-\rho} \log |f(t_k z)| \qquad \text{and} \qquad v(z) = \lim_{k\to\infty} t_k^{-\rho} \log |(f \star H)(t_k z)|$$

exist, and (23) implies that $v \leqslant u$, by a similar argument to that by which (19) was derived. Now the equation

$$f(z) = 1 + ze^{2\pi i\alpha}(f \star H)(ze^{i\beta}) \tag{26}$$

implies that

$$u(z) \leqslant \max\{0, v(ze^{i\beta})\} \leqslant \max\{0, u(ze^{i\beta})\},$$

and as β is irrational, we conclude that u does not depend on $\arg z$.

This completes the proof. □

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